

Supplement to Linear Multistep Methods for Volterra Integral and Integro-Differential Equations

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In these appendices we present, successively,

- I conditions for the existence of a unique solution of (1.1) and (1.2);
- II three tables of coefficients of forward differentiation formulas, and of two common LM formulas for ODEs, viz., backward differentiation formulas and Adams-Moulton formulas;
- III two lemmas which are needed in:
- IV proofs of the main results of this paper, as far as they are non-trivial (in the opinion of the authors).

APPENDIX I

Conditions for the existence of a unique solution $y(t) \in C(I)$ of (1.1) with $\theta = 1$

- $K(t, \tau, y)$ is continuous with respect to t and τ , for all $(t, \tau) \in S$;
- K satisfies a (uniform) Lipschitz condition with respect to y , i.e.,
 $|K(t, \tau, y) - K(t, \tau, z)| \leq L_1 |y - z|$, for all $(t, \tau) \in S$, for all finite $y, z \in \mathbb{R}$;
- $g(t) \in C(I)$. \square

Conditions for the existence of a unique solution $y(t) \in C(I)$ of (1.1) with $\theta = 0$

- $K(t, \tau, y) \in C^1(S \times \mathbb{R})$;
- for $t = \tau$ the derivative $\partial K / \partial y$ is bounded away from zero:
 $|\partial K(t, \tau, y) / \partial y| \geq r_0 > 0$ for all $t \in I, y \in \mathbb{R}$;
- $\partial K(t, \tau, y) / \partial t$ satisfies a (uniform) Lipschitz condition with respect to y on $S \times \mathbb{R}$;
- $g(t) \in C^1(I)$ with $g(t_0) = 0$. \square

Conditions for the existence of a unique solution $y(t) \in C^1(I)$ of (1.2), for given initial value $y(t_0) = y_0$

The following three (uniform) Lipschitz conditions:

- $|f(t, y_1, z) - f(t, y_2, z)| \leq L_1 |y_1 - y_2|$, for all $t \in I$, for all finite $z, y_1, y_2 \in \mathbb{R}$;

Table 3 Coefficients of the Adams-Bashforth formulas

for ODEs $f'(t) = g(t) : f_n - f_{n-1} = \sum_{i=0}^k b_i g_{n-i}$

k	b ₀	b ₁	b ₂	b ₃	b ₄	b ₅
1	1/2		1/2			
2	5/12	2/3	-1/12			
3	3/8	19/24	-5/24	1/24		
4	251/720	323/360	-11/30	53/360	-19/720	
5	95/288	1427/1440	-133/240	241/720	-137/1440	3/160

APPENDIX III

LEMMA A.1. Let $z \geq 0$ for $n = 0, 1, \dots, N$, and suppose that

$$z_n \leq hc \sum_{i=0}^{n-1} z_i + C_2, \quad n = k, k+1, \dots, N,$$

where $k > 0, h > 0$ and $C_1 > 0$ ($i=1, 2$). Suppose, moreover, that $z_j \leq z/k$ for $j = 0, 1, \dots, k-1$. Then

$$z_n \leq (hc_1 z + C_2)(1 + hc_1)^{n-k}, \quad n = k, k+1, \dots, N.$$

PROOF. See [7].

LEMMA A.2. Consider the linear inhomogeneous difference equation with constant coefficients τ_j :

$$(A.1) \quad \tau_{n+k} + \tau_{n+k-1} + \dots + \tau_{n+1} = g_{n+k}, \quad n \geq 0,$$

where $\{g_n\}$ is a given sequence, independent of the y_i .
 (i) If the characteristic polynomial $\tau(z) := \sum_{j=0}^k \tau_j z^{k-j}$ is simple von Neumann (cf. Section 2.3) then the solution of (A.1) satisfies the inequality

- $|f(t, y, z_1) - f(t, y, z_2)| \leq L_2 |z_1 - z_2|$, for all $t \in I$, for all finite $y_1, z_1, z_2 \in \mathbb{R}$;
 - $|K(t, \tau, y_1) - K(t, \tau, y_2)| \leq L_3 |y_1 - y_2|$, for all $(t, \tau) \in S$, for all finite $y_1, y_2 \in \mathbb{R}$. \square

APPENDIX II

Table 1 Coefficients of forward differentiation formulas

$$f'(t) \approx \sum_{\ell=0}^{k-1} \delta_\ell f(t + \ell h), \quad t_{n+\ell} = t_n + \ell h$$

k	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5
1	1					
2	3/2	-2	1/2			
3	11/6	-3	3/2	-1/3		
4	25/12	-4	3	-4/3	1/4	
5	137/60	-5	5	-10/3	5/4	-1/5

Table 2 Coefficients of the backward differentiation formulas

$$\text{for ODEs } f'(t) = g(t) : \sum_{i=0}^k a_i f_{n-i} = b_0 g_n$$

k	a ₀	a ₁	a ₂	a ₃	a ₄	a ₅	b ₀
1	1	-1					1
2	1	-4/3	1/3				2/3
3	1	-18/11	9/11	-2/11			6/11
4	1	-48/25	36/25	-16/25	3/25		12/25
5	1	-300/137	300/137	-200/137	75/137	-12/137	60/137

where $(-i)^{\ell-1}$ is assumed to be zero for $i = \ell = 0$. Equating to zero all terms in the $\sum_{\ell=0}^q$ yields the order equations (2.2,3) and at the same time $L_n(Y) = O(h^{p+1})$ as required in Definition 2.2.1. \square

PROOF OF THEOREM 2.2.2. Taylor expansion of $Y(t_{n+j}, \tau_{n-1})$ around (t_n, τ_n) yields

$$Y(t_{n+j}, \tau_{n-1}) = \sum_{q=0}^p \frac{1}{q!} h^q \left[j \frac{\partial}{\partial t} - i \frac{\partial}{\partial s} \right]^q Y(t, s) \Big|_{(t_n, \tau_n)} + O(h^{p+1}) \text{ as } h \rightarrow 0.$$

In order to exploit the fact that $Y(t, t) \equiv 0$ (see definition 2.2.1), we introduce the variables $u = t + s$ and $v = t - s$ and write

$$Y(t, s) = Y\left(\frac{u+v}{2}, \frac{u-v}{2}\right) =: Z(u, v).$$

The identity $Y(t, t) \equiv 0$ implies that Z and all its derivatives with respect to u vanish for $u = 2t$ and $v = 0$. In the following we use the notation

$$Z(n, m) := \frac{\partial^{n+m} Z}{\partial u^n \partial v^m}(2t_n, 0).$$

By means of the binomial theorem we have

$$(A.2) \quad Y(t_{n+j}, \tau_{n-1}) = \sum_{q=0}^p \frac{1}{q!} h^q [(j-i) \frac{\partial}{\partial u} + (j+i) \frac{\partial}{\partial v}]^q Z(u, v) \Big|_{(2t_n, 0)} + O(h^{p+1}) \\ = \sum_{q=0}^p \sum_{\ell=0}^q \frac{1}{\ell!} h^\ell \binom{q}{\ell} (j-i)^{\ell-q} (j+i)^q Z^{(\ell, q-\ell)} + O(h^{p+1}) \text{ as } h \rightarrow 0$$

and

$$|y_n| \leq C \left(\max_{0 \leq j \leq k-1} |y_j| + \sum_{j=k}^n |g_j| \right), \quad n \geq k,$$

where C is independent of n .

(ii) If $\zeta(x)$ is Schur (cf. Section 2.3) then the solution of (A.1) satisfies the inequality

$$|y_n| \leq C \left(\max_{0 \leq j \leq k-1} |y_j| + \max_{k \leq j \leq n} |g_j| \right), \quad n \geq k,$$

where C is independent of n .

PROOF. See [7].

APPENDIX IV

PROOF OF THEOREM 2.2.1. Taylor expansion of $Y(t_{n+j}, \tau_{n-1})$ around (t_n, τ_n) yields

$$L_n[Y] = \sum_{i=0}^k \left\{ \alpha_i \sum_{q=0}^p \frac{1}{q!} h^q (-i) \frac{\partial}{\partial t} - i \frac{\partial}{\partial s} \right\}^q Y(t, s) \\ + \sum_{j=-k}^k \left\{ \beta_j - \gamma_j \frac{\partial}{\partial s} \right\} \sum_{q=0}^p \frac{1}{q!} h^q \left(j \frac{\partial}{\partial t} - i \frac{\partial}{\partial s} \right)^q Y(t, s) \Big|_{(t_n, \tau_n)} \\ + O(h^{p+1}) \text{ as } h \rightarrow 0.$$

Writing this formula in the form

$$L_n[Y] = \sum_{i=0}^p \frac{1}{i!} h^i (D_i Y) \Big|_{(t_n, \tau_n)} + O(h^{p+1})$$

and expanding the differential operator D_i by the binomial theorem we find

$$D_i = \sum_{k=0}^i \alpha_k \left\{ (-i) \frac{\partial}{\partial t} - i \frac{\partial}{\partial s} \right\}^k + \sum_{j=-k}^k \left[j^k \beta_j \frac{\partial}{\partial t} - (i \beta_j + \alpha_j \gamma_j) \frac{\partial}{\partial s} \right] j^k \frac{\partial}{\partial t} - i \frac{\partial}{\partial s} \Big\}^k \\ = \sum_{\ell=0}^q \sum_{i=0}^k \left\{ (-i) \right\}^q \alpha_i \sum_{j=-k}^k j^q \ell (-i)^{\ell-1} \left[i \beta_j + \alpha_j \gamma_j \right] \binom{q}{\ell} \frac{\partial}{\partial t}^{\ell} \frac{\partial}{\partial s}^{\ell},$$

From these expansions it is immediate that the VLM formula (2.1.4) satisfies the relation

$$(A.4) \quad \sum_{i=0}^k \left\{ \sum_{j=-k}^i y_j(t_{n-i}) + \sum_{j=-k}^i \left[\beta_{ij} Y_{n-i}(t_{n+j}) - h Y_{ij}^K(t_{n+j}) \right] \right\} \\ = \sum_{q=0}^m h \left\{ \frac{\partial^q}{\partial t^q} \left(\frac{\partial Y}{\partial t} \right) (t_n) + \sum_{\ell=0}^q (C_{q\ell} - \frac{A_{q\ell}}{E_1(q-\ell)}) \frac{\partial^{q-\ell}}{\partial t^{q-\ell}} Y(t_n, t_n) \right\} \\ + O(h^{m+1})$$

where $A_{q\ell}$ and $C_{q\ell}$ are defined by (2.3.2) and (2.2.3), respectively. Under the conditions of the theorem it is easily verified that this equation leads to (2.3.3). Furthermore, (2.3.3) is obviously the m -times differentiated form of equation (1.1). \square

PROOF OF THEOREM 2.3.2. Let $Y(t, s)$ be given by (1.6) where $y(t)$ is the exact solution of (1.1), then we may write for $n \geq k$

$$L_n(Y) = L_n(Y) - \sum_{i=0}^k [\alpha_i Y_{n-i} + \sum_{j=-k}^i (\beta_{ij} Y_{n-i}(t_{n+j}) - h Y_{ij}^K(t_{n+j}))] \\ = \sum_{i=0}^k \left\{ \alpha_i t_{n-i} + \sum_{j=-k}^i [\beta_{ij} (Y(t_{n+j}, t_{n-i}) - Y_{n-i}(t_{n+j}))] \right\} \\ - h Y_{ij}^K(t_{n+j}, t_{n-i}, Y(t_{n-i})) - K_{n-i}(t_{n+j}) \}.$$

Substitution of the functions $Y(t, s)$ and $Y_n(t)$ and using (2.1.3) and (2.3.6b) leads to

$$(A3) \quad h Y_{q\ell}(t_{n+j}, t_{n-i}) = \sum_{q=0}^p \sum_{\ell=0}^q \frac{1}{q!} h^{q+1} \binom{q}{\ell} (j-i)^{q-\ell} (j+i)^{\ell} [Z^{q-\ell+1, \ell} - Z^{q-\ell, \ell+1}] \\ + O(h^{p+1}) \\ = \sum_{q=0}^p \sum_{\ell=0}^q \frac{1}{q!} h^q \binom{q}{\ell} (j-i)^{q-\ell-1} (j+i)^{\ell-1} [(q+j+q-2\ell)Z^{q-\ell, \ell} \\ + O(h^{p+1})] \text{ as } h \rightarrow 0.$$

Substitution of (A.2) and (A.3) into $L_n[Y]$ and using $Z^{(q, 0)} = 0$ yields

$$L_n[Y] = \sum_{q=1}^p \frac{1}{q!} h^q \sum_{\ell=1}^q \binom{q}{\ell} B_{q\ell} Z^{(q-\ell, \ell)} + O(h^{p+1}) \text{ as } h \rightarrow 0$$

where $B_{q\ell}$ is defined in (2.2.4). This proves the theorem. \square

PROOF OF THEOREM 2.3.1.

PROOF. Taylor expansion in a fixed point $t = t_n$ yields, respectively,

$$y(t_{n-i}) = \sum_{q=0}^m \frac{1}{q!} (-i) \frac{d^q}{dt^q} y(t_n) + O(h^{m+1}),$$

$$Y_{n-i}(t_{n+j}) = Y(t_{n+j}, t_{n-i}) - E_{n-i}(h; t_{n+j}) \\ = \sum_{q=0}^m \frac{1}{q!} h^q (j-2i) \frac{d^q}{dt^q} Y(t_n, t_n) + O(h^{m+1}) \\ = \sum_{q=0}^m \frac{1}{q!} h^q \sum_{\ell=0}^q j^{q-\ell} (-i)^{\ell} \frac{\partial^{q-\ell}}{\partial t^{q-\ell}} \frac{\partial^{\ell}}{\partial s^{\ell}} Y(t_n, t_n) \\ + O(h^{m+1})$$

$$K_{n-i}(t_{n+j}) = K(t_{n+j}, t_{n-i}, Y(t_{n-i})) = \frac{\partial}{\partial s} Y(t_{n+j}, t_{n-i}) \\ = \sum_{q=0}^m \frac{1}{q!} h^{q-1} \sum_{\ell=0}^q j^{q-\ell} (-i)^{\ell} \frac{\partial^{q-\ell}}{\partial t^{q-\ell}} \frac{\partial^{\ell}}{\partial s^{\ell}} Y(t_n, t_n) \\ + O(h^m).$$

$$(A.5) \quad L_n(Y) = \sum_{i=0}^k \alpha_i \epsilon_{n-i} + \sum_{j=k}^n [\beta_{ij} \left(h \sum_{\ell=0}^{n-i} w_{n-i, \ell} \Delta K(t_{n+j}, t_{\ell}, y(t_{\ell}), y_{\ell}) \right. \\ \left. + E_{n-i}(h; t_{n+j}) \right) - h \gamma_{ij} \Delta K(t_{n+j}, t_{n-i}, y(t_{n-i}), y_{n-i})]$$

Thus, we have found for the errors ϵ_n the relation

$$(A.6) \quad \sum_{i=0}^k \alpha_i \epsilon_{n-i} = v_n, \quad n \geq k^*, \text{ where} \\ v_n = L_n(Y) - \sum_{i=0}^k \left[h \beta_{ij} \sum_{\ell=0}^n w_{n-i, \ell} \Delta K(t_{n+j}, t_{\ell}, y(t_{\ell}), y_{\ell}) \right. \\ \left. + \beta_{ij} E_{n-i}(h; t_{n+j}) \right] - h \gamma_{ij} \Delta K(t_{n+j}, t_{n-i}, y(t_{n-i}), y_{n-i})$$

We now proceed with the two cases (a) and (b) separately.

$$(a) \quad \alpha(z) \equiv \alpha_0 z^k, \quad \alpha_0 \neq 0.$$

We want to apply the discrete Gronwall inequality stated in Lemma A.1 in order to derive an upper bound for the solution of this linear difference equation, and therefore we need an upper bound for $|v_n|$. A straightforward calculation yields

$$(A.7) \quad |v_n| \leq T(h) + \sum_{i=0}^k \sum_{j=n-k}^n [b w_{i, j} h \sum_{\ell=0}^n | \epsilon_{\ell} | + c_{1, j} h | \epsilon_{n-i} | + b E(h)] \\ \leq C_0 h \sum_{\ell=0}^n | \epsilon_{\ell} | + C_1 E(h) + T(h),$$

where C_0 and C_1 are constants independent of h and n (in the following all constants C_j will be independent of h and n). From (A.6) it follows that

$$| \epsilon_n | | \epsilon_n | \leq C_0 h \sum_{\ell=0}^n | \epsilon_{\ell} | + C_1 E(h) + T(h)$$

so that for h sufficiently small

$$| \epsilon_n | \leq \frac{1}{| \alpha_0 |^{-1} C_0 h} [C_0 h \sum_{\ell=0}^{n-1} | \epsilon_{\ell} | + C_1 E(h) + T(h)] \\ \leq C_2 h \sum_{\ell=0}^{n-1} | \epsilon_{\ell} | + C_3 [E(h) + T(h)].$$

Application of Lemma A.1 (with $z = k^*(h)$) yields

$$| \epsilon_n | \leq (1 + C_2 h)^{n-k^*} (k^* h C_2 \delta(h) + C_3 [E(h) + T(h)]), \\ n = k^*, \dots, N.$$

Since $n h \leq T - t_0$, part (a) of the theorem is immediate.

$$(b) \quad \alpha(z) \text{ is simple von Neumann, } \beta(z) \equiv 0.$$

Instead of directly applying Lemma A.1 to the inequality (obtained from (A.6))

$$\sum_{i=0}^k | \alpha_i | | \epsilon_{n-i} | \leq | v_n |,$$

we first apply Lemma A.2 (i) to obtain the "sharper" inequality

$$(A.8) \quad | \epsilon_n | \leq C_0 \delta(h) + \sum_{j=k}^n | v_j |, \quad n \geq k^*.$$

Unfortunately, if we use the upper bound (A.7) for $|v_j|$ and then apply Lemma A.1, we cannot prove convergence. However, by using the property $\beta(z) \equiv 0$, that is $\beta_i = \sum_{j=n-k}^k \beta_{ij} = 0$, a sharper upper bound than (A.7) can be derived. To that end we write

Since $nh \leq T - t_0$ we find for h sufficiently small

$$|e_n| \leq C_3 h \sum_{\ell=0}^{n-1} |\epsilon_\ell| + C_6 h^{-1} [h\delta(h) + \Delta E(h) + T(h)].$$

Finally, by applying Lemma A.1 we arrive at the estimate

$$|e_n| \leq (1+C_5 h)^{n-k} \left(k h C_5 \delta(h) + C_6 h^{-1} [h\delta(h) + \Delta E(h) + T(h)] \right),$$

from which part (b) of the theorem follows. \square

PROOF OF THEOREM 2.3.4. Following the first lines of the proof of Theorem 2.3.2 we obtain the following relation, analogous to (A.5), where

$$K_{rs} := K(t_r, t_s)$$

$$(A.10) \quad \sum_{i=0}^k \sum_{j=-k}^k \gamma_{ij} K_{n+j, n-i} \epsilon_{n-i} = \sum_{i=0}^k \sum_{j=-k}^k \beta_{ij} \left[\sum_{\ell=0}^k \omega_{n-i, \ell} K_{n+j, \ell} \epsilon_{j+\ell} + h^{-1} E_{n-i}(h; t_{n+j}) \right] - h^{-1} L_{-k_n}(Y), \quad n \geq k^*.$$

Now we write $K_{n+j, n-i} = K_{nn} + (K_{n+j, n-i} - K_{nn})$ and $K_{n+j, \ell} = K_{n\ell} + (K_{n+j, \ell} - K_{n\ell})$ and rewrite (A.10) to obtain

$$(A.11) \quad \sum_{i=0}^k \sum_{j=-k}^k \gamma_{ij} \epsilon_{n-i} = v_n, \quad n \geq k^*,$$

where

$$K_{nn} v_n = h \sum_{i,j} \gamma_{ij} \left(\frac{K_{nn} - K_{n+j, n-i}}{h} \right) \epsilon_{n-i} + \sum_{i,j} \beta_{ij} \sum_{\ell} \omega_{n-i, \ell} K_{n\ell} \epsilon_\ell + h \sum_{i,j} \beta_{ij} \sum_{\ell} \omega_{n-i, \ell} \left(\frac{K_{n+j, \ell} - K_{n\ell}}{h} \right) \epsilon_\ell + h^{-1} \sum_{i,j} \beta_{ij} E_{n-i}(h; t_{n+j}) - h^{-1} L_{-k_n}(Y).$$

$$(A.13) \quad L_{-k_n}^*(Y) = \sum_{i=0}^k \sum_{j=-k}^k [\alpha_{ij}^* \epsilon_{n-i} - h \gamma_{ij}^* \Delta E_{n-i}],$$

$$\begin{aligned} & \left| \sum_{j=-k}^k \beta_{ij} \Delta K(t_{n+j}, t_{n-i}, \tau_\ell, \nu_\ell) \right| = \sum_{j=-k}^k \beta_{ij} [\Delta K(t_{n+i}, t_{n-i}, \tau_\ell, \nu_\ell) + \Delta K(t_{n+j}, t_{n-i}, \tau_\ell, \nu_\ell) - \Delta K(t_{n+i}, t_{n-i}, \tau_\ell, \nu_\ell)] \\ & \leq b \sum_{j=-k}^k |\beta_{ij}| |\nu_\ell|, \end{aligned}$$

and, similarly,

$$\left| \sum_{j=-k}^k \beta_{ij} E_{n-i}(h; t_{n+j}) \right| \leq b \sum_{j=-k}^k \Delta E(h).$$

In this way we obtain instead of (A.7) the upper bound

$$(A.9) \quad \begin{aligned} |v_n| & \leq T_n(h) + \sum_{i=0}^k \sum_{j=-k}^k [b \omega_{i,j} |h|^2 \sum_{\ell=0}^n |\epsilon_\ell| + c L_n |h| \sum_{\ell=0}^n |\epsilon_\ell| + b \Delta E(h)] \\ & \leq C_1 h \sum_{i=0}^k [|\epsilon_{n-i}| + h \sum_{\ell=0}^n |\epsilon_\ell|] + C_2 \Delta E(h) + T(h). \end{aligned}$$

Substitution into (A.8) yields the inequality

$$|e_n| \leq C_3 \left\{ \delta(h) + h \sum_{j=k+1}^n |\epsilon_{j-1}| + h \sum_{\ell=0}^j |\epsilon_\ell| + h^{-1} \Delta E(h) + h^{-1} T(h) \right\}.$$

It is easily verified that

$$\sum_{j=k}^n |\epsilon_{j-1}| \leq (k+1) \sum_{j=0}^n |\epsilon_j|.$$

Hence,

$$|e_n| \leq C_4 \left\{ \delta(h) + h \left[\sum_{\ell=0}^{n+1} |\epsilon_\ell| + nh^{-1} \Delta E(h) + nh^{-1} T(h) \right] \right\}.$$

Since $\gamma(z)$ is Schur, we may apply Lemma A.2 (ii) to (A.11) and find

$$(A.12) \quad |e_n| \leq C(\delta(h) + \max_{k < i < n} |\nu_i|), \quad n \geq k^*.$$

$$(A.14) \quad \sum_{i=0}^k a_i \varepsilon_i = v_n^*$$

where v_n^* satisfies the inequality (using (1.31') and (1.31''))

$$|v_n^*| := |L_n^*[y] + h \sum_{i=0}^k \gamma_i^* \Delta f_{n-i}| \leq T_n^*(h) + h \sum_{i=0}^k |\gamma_i^*| [L_1 | \varepsilon_{n-i} | + L_2 | \eta_{n-i} |].$$

Application of Lemma A.2 (i) yields (because $\alpha^*(z)$ is simple von Neumann)

$$(A.15) \quad |\varepsilon_n| \leq C_0 \left[h \sum_{j=0}^n |\varepsilon_j| + |\eta_j| \right] + \delta(h) + \sum_{j=k}^n T_j^*(h)$$

where C_0 is some constant independent of n and h .

For n we derive from the second relation in (A.13)

$$(A.16) \quad \sum_{i=0}^k \alpha_i \eta_{n-i} = v_n$$

where v_n is defined as in (A.6).

(a) In the case where $\alpha(z) = \alpha_0 z^k$ we have from (A.7):

$$|\eta_n| \leq C_1 [E_n(h) + h \sum_{\ell=0}^n |\varepsilon_\ell| + T_n(h)], \quad n \geq k$$

$$\begin{aligned} & + C_2 h \omega \sum_{i,j} |\beta_{ij}| \sum_{\ell=0}^k |\varepsilon_\ell| + h^{-1} \sum_{i,j} \beta_{ij} E_{n-i}(h; \tau_{rsj}) + h^{-1} |L_T(Y)|, \\ & \dots \dots \dots \quad i=0 \dots \ell=0 \dots \dots \dots \quad r \geq k. \end{aligned}$$

Now we use the condition $\beta(z) \equiv 0$, i.e., $\beta_i = 0$, and (2.3.6a) to obtain (cf. the derivation of (A.9) in the proof of Theorem 2.3.2)

$$|v_r| \leq C_3 \left\{ h \sum_{i=0}^k |\varepsilon_{r-i}| + h \sum_{\ell=0}^r |\varepsilon_\ell| + h^{-1} \Delta E(h) \right\} + h^{-1} T(h), \quad r \geq k,$$

$$\leq C_4 \left\{ h \sum_{\ell=0}^r |\varepsilon_\ell| + h^{-1} \Delta E(h) \right\} + h^{-1} T(h).$$

Substituting this into (A.12) we find, for h sufficiently small,

$$|\varepsilon_n| \leq C_5 \left\{ \delta(h) + h^{-1} \Delta E(h) + h^{-1} T(h) + h \sum_{\ell=0}^{n-1} |\varepsilon_\ell| \right\}$$

and application of lemma A.1 yields the result of the theorem. \square

PROOF OF THEOREM 3.3.1. Proceeding as in the proof of Theorem 2.3.2 we derive the relations

for some constant C_1 . Substitution into (A.15) yields

$$\begin{aligned}
 |c_n| &\leq C_2 \left\{ h \sum_{j=k}^n [|\epsilon_j| + h \sum_{\ell=0}^j |\epsilon_\ell| + E_j(h) + \right. \\
 &\quad \left. + T_j(h) + h^{-1} T_j^*(h)] + \delta(h) + h\delta^*(h) \right\} \\
 &\leq C_3 \left\{ h \sum_{j=0}^n |\epsilon_j| + E_n(h) + T_n(h) + h^{-1} T_n^*(h) + \delta(h) + h\delta^*(h) \right\}
 \end{aligned}$$

where we have used that $nh \in T - \tau_0$. From Lemma A.1, part (a) of the theorem easily follows.

(b) Since $\alpha(z)$ is simple von Neumann, we apply Lemma A.2 (i) to (A.16) and use (A.9) (since $\beta(z) \equiv 0$) to find

$$\begin{aligned}
 |r_n| &\leq C_4 \left\{ \delta^*(h) + \sum_{j=k-i}^n \prod_{i=0}^k (h |\epsilon_{j-i}| + h^2 \sum_{\ell=0}^j |\epsilon_\ell|) + \Delta E_j(h) + T_j(h) \right\} \\
 &\leq C_5 \left\{ h \sum_{j=0}^n |\epsilon_j| + \delta^*(h) + h^{-1} \Delta E_n(h) + h^{-1} T_n(h) \right\}.
 \end{aligned}$$

Substitution into (A.15) and applying Lemma A.1 leads to part (b) of the theorem. \square

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